Solution 3

1. A bounded function f on $[a,b]$ is said to be locally Lipschitz continuous at $x \in [a,b]$ if there exist some L and δ such that

$$|f(y) - f(x)| \le L|x-y| \ , \quad \forall y \in (x-\delta, x+\delta).$$

Show that f is Lipschitz continuous at x.

Solution. For y lying outside $(x-\delta, x+\delta)$, $|y-x| \ge \delta$. Therefore,

$$|f(y)-f(x)| = \frac{|f(y)-f(x)|}{|y-x|}\ |y-x| \le \frac{2\|f\|_\infty}{\delta}|y-x| \ .$$

Hence.

$$|f(y)-f(x)| \le L'(|y-x|, \quad \forall y \ , \quad L' = \max\{L, 2\|f\|_\infty/\delta\} \ .$$

2. Let f be a function defined on (a,b) and $x_0 \in (a,b)$.

(a) Show that f is Lipschitz continuous at x_0 if its left and right derivatives exist at x_0.

(b) Construct a function Lipschitz continuous at x_0 whose one sided derivatives do not exist.

Solution. (a) Let $\alpha = f'_+(x_0)$ and $\beta = f'_-(x_0)$. For $\varepsilon = 1 > 0$, there exists δ_1 such that

$$\left|\frac{f(x+z)-f(x)}{z} - \alpha\right| < 1 \ ,$$

for $0 < z < \delta_1$. It follows that

$$|f(x+z)-f(x)| \le |f(x+z)-f(x)-\alpha z| + |\alpha z| \le (1+|\alpha|)|z| \ .$$

Similarly,

$$|f(x+z)-f(x)| \le (1+|\beta|)|z| \ , \quad z \in (-\delta_2, 0) \ .$$

We conclude that $|f(x+z)-f(x)| \le (1+\gamma)|z|, \quad z \in (-\delta,\delta), \ \delta = \min\{\delta_1,\delta_2\} \ , \gamma = \max\{|\alpha|,|\beta|\}$. By Problem 1, it is Lipschitz continuous at x_0.

(b) The function $f(x) = x\sin\frac{1}{x}$ $(x \ne 0)$ and $= 0$ at $x = 0$. It is Lipschitz continuous at $x_0 = 0$ with $L = 1$ but both one-sided derivatives do not exist.

3. Can you find a cosine series which converges uniformly to the sine function on $[0,\pi]$? If yes, find one.

Solution. Yes, extend the sine function on $[0,\pi]$ to $|\sin x|$, an even, 2π-periodic function. Since it is continuous, piecewise C^1, its cosine series converges uniformly to this extended function. In particular, this cosine series converges uniformly to $\sin x$ on $[0,\pi]$.

4. A sequence $\{a_n\}, n \ge 0$, is said to converge to a in mean if

$$\frac{a_0 + a_1 + \cdots + a_n}{n+1} \to a \ , \quad n \to \infty \ .$$

(a) Show that $\{a_n\}$ converges to a in mean if $\{a_n\}$ converges to a.

(b) Give a divergent sequence which converges in mean.

Solution. (a) For $\varepsilon > 0$, there is some n_0 such that $|a_n - a| < \varepsilon$ for all $n > n_0$. Now

$$\begin{aligned}
\left|\frac{a_0 + \cdots + a_n}{n+1} - a\right| &= \left|\frac{a_0 + \cdots + a_{n_0}}{n+1} + \frac{a_{n_0+1} + \cdots + a_n}{n+1} - a\right| \\
&= \left|\frac{a_0 + \cdots + a_{n_0}}{n+1} + \frac{(a_{n_0+1} - a) + \cdots + (a_n - a)}{n+1} - \frac{n_0+1}{n+1}a\right| \\
&\le \frac{n - n_0}{n+1}\varepsilon + \frac{a_0 + \cdots + a_{n_0}}{n+1} + \frac{n_0+1}{n+1}a \\
&\le 2\varepsilon,
\end{aligned}$$

after taking $n \ge n_1$ for a much larger n_1.

(b) Consider the sequence $\{(-1)^n\}$.

5. Let D_n be the Dirichlet kernel and define the Fejer kernel to be $F_n(x) = \dfrac{1}{n+1}\sum_{k=0}^n D_k(x)$.

   (a) Show that

   $$F_n(x) = \frac{1}{2\pi(n+1)}\left(\frac{\sin(\frac{n+1}{2})x}{\sin x/2}\right)^2, \quad x \ne 0 .$$

   (b) Let

   $$\sigma_n f(x) = \frac{1}{n+1}\sum_{k=0}^n S_k f(x) .$$

   Show that for every $x \in [-\pi, \pi]$, $\sigma_n f(x)$ converges uniformly to $f(x)$ for any continuous, 2π-periodic function f. Hint: Follow the proof of Theorem 1.5 and use the non-negativity of F_n.

Solution. (a) Use $2\sin z/2 \sin(k/2+1)z = \cos kz - \cos(k+1)z$ and $1 - \cos(n+1)z = 2\sin^2 \frac{n+1}{2}z$ to get it.

(b) Note that $\int_{-\pi}^{\pi} F_n(z)\,dz = 1$ as it holds for D_n. Proceeding as in the proof of Theorem 1.5,

$$\begin{aligned}
(\sigma_n f)(x) - f(x) &= \int_{-\pi}^{\pi} F_n(z)(f(x+z) - f(x)\,dx \\
&= \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi} \Phi_\delta(z)\frac{\sin^2(\frac{n+1}{2}z)}{\sin^2 z/2}(f(x+z) - f(x))\,dz \\
&\quad + \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi} (1 - \Phi_\delta(z))\frac{\sin^2(\frac{n+1}{2}z)}{\sin^2 z/2}(f(x+z) - f(x))\,dz \\
&= I + II .
\end{aligned}$$

For the first term, for $\varepsilon > 0$, there is some δ such that $|f(y) - f(x)| < \varepsilon$, for $y, |y - x| < \delta$. Thus,

$$\begin{aligned}
|I| &\le \left|\int_{-\delta}^{\delta} \Phi_\delta(z)F_n(z)(f(x+z) - f(x))\,dz\right| \\
&\le \varepsilon\int_{-\delta}^{\delta} F_n(z)\,dz \\
&\le \varepsilon\int_{-\pi}^{\pi} F_n(z)\,dz \\
&= \varepsilon .
\end{aligned}$$

The second is easy to handle: For this fixed δ,

$$|II| \le \frac{1}{2\pi(n+1)} \times \frac{1}{\sin^2 \delta/4} \times 2\|f\|_\infty \to 0 \ ,$$

as $n \to \infty$.

6. Let f and g be two continuous, 2π-periodic functions whose Fourier series are the same. Prove that $f \equiv g$.

   Solution. Replacing $f-g$ by a single f, it suffices to show that $f \equiv 0$ if its Fourier series vanishes identically. Indeed, from $\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = 0$ for all $n \in \mathbb{Z}$ we know that

   $$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \ ,$$

   for all finite trigonometric series g. As f is continuous, by Weierstrass approximation theorem, for every $\varepsilon > 0$, there is a such g satisfying $\|f-g\|_\infty < \varepsilon$. Therefore,

   $$\begin{aligned} \left|\int_{-\pi}^{\pi} f^2(x)\,dx\right| &\le \left|\int_{-\pi}^{\pi} f(x)(f(x)-g(x))\,dx\right| + \left|\int_{-\pi}^{\pi} f(x)g(x)\,dx\right| \\ &\le \varepsilon \int_{-\pi}^{\pi} |f(x)|\,dx \ . \end{aligned}$$

   Since $\varepsilon > 0$ can be arbitrarily small, $\int_{-\pi}^{\pi} f^2 = 0$, so $f \equiv 0$.

7. Let $f, g \in R_{2\pi}$ whose Fourier series are the same. Show that $\int_{-\pi}^{\pi}(f-g)^2(x)\,dx = 0$ and conclude that f and g are equal almost everywhere.

   Solution. Replacing $f-g$ by a single f, it suffices to show that $\int_{-\pi}^{\pi} f^2 = 0$ if its Fourier series vanishes identically. Indeed, as in the previous problem, $\int_{-\pi}^{\pi} fg = 0$ for all finite trigo series g. By Weierstrass approximation theorem, this relation holds for all continuous g. Now, for a characteristic function $\chi_{[a,b]}$ we consider the continuous piecewise linear function g which is equal to 1 on $[a+\delta, b-\delta]$ and vanishes outside $[a,b]$. Then, from

   $$\int_{-\pi}^{\pi} f\chi_{[a,b]}\,dx = \int_{-\pi}^{\pi} f(\chi_{[a,b]}-g)\,dx + \int_{-\pi}^{\pi} fg\,dx = \int_{-\pi}^{\pi} f(\chi_{[a,b]}-g)\,dx \ ,$$

   we get

   $$\left|\int_{-\pi}^{\pi} f\chi_{[a,b]}\,dx\right| \le \int_{-\pi}^{\pi} |f||\chi_{[a,b]}-g|\,dx \le 2\|f\|_\infty\delta.$$

   Since $\delta > 0$ is arbitrarily small, we conclude $\int_{-\pi}^{\pi} f\chi_{[a,b]}\,dx = 0$. It follows that $\int_{-\pi}^{\pi} fs\,dx = 0$ for all step functions. Choosing the "Darboux lower sum" sequence $s(x) = \sum_j m_j\chi_{I_j}(x)$ which approximate f from below and satisfy

   $$\int_{-\pi}^{\pi} (f-s)\,dx \to 0 \ ,$$

   we have

   $$\begin{aligned} \int_{-\pi}^{\pi} f^2\,dx &= \left|\int_{-\pi}^{\pi} f(f-s)\,dx + \int_{-\pi}^{\pi} fs\,dx\right| \\ &= \left|\int_{-\pi}^{\pi} f(f-s)\,ds\right| \\ &\le \|f\|_\infty \left|\int_{-\pi}^{\pi} (f-s)\,dx\right| \\ &\to 0 \ . \end{aligned}$$

Note. Problems 6 and 7 provide a proof to the uniqueness theorem stated in Section 1.2 in our lecture notes.